Entropic metastability in the narrow escape problem

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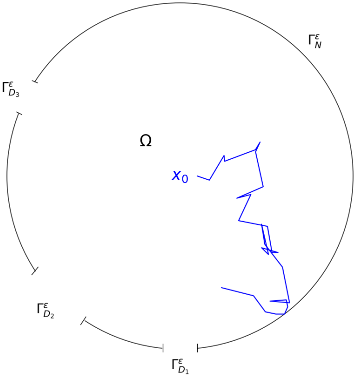
The narrow escape problem

Rare event: metastability induced by entropic barriers ($\neq$ energetic)



Setting:

- Domain Ω with holes Γ^{ε}_{D} and reflecting boundary Γ^{ε}_{N}
- A Brownian motion starting at x_0 taking a long time to exit $\tau_\varepsilon = \inf\{t \geq 0 \,|\, X_t \notin \overline{\Omega}\}$

$$\mathrm{d}X_t = \sqrt{2}\,\mathrm{d}B_t - \mathbb{1}_{\Gamma^{\varepsilon}_{\mathrm{D}}}(X_t)n(X_t)\mathrm{d}L_t$$

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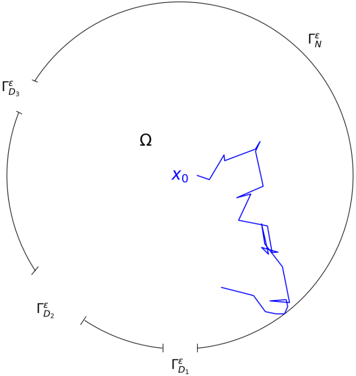

Setting:

- Domain Ω with holes Γ_D^ε and reflecting boundary Γ_N^ε
- A Brownian motion starting at x_0 taking a long time to exit $\tau_\varepsilon = \inf\{t \geq 0 \,|\, X_t \notin \overline{\Omega}\}$

$$\mathrm{d}X_t = \sqrt{2}\,\mathrm{d}B_t - \mathbb{1}_{\Gamma_{\mathrm{D}}^\varepsilon}(X_t)n(X_t)\mathrm{d}L_t$$

Goal: In the limit of small holes $\varepsilon \to 0$:

- The law of the escape time τ_ε
- The law of exit hole X_{τ_ε}

Motivation: Entropic transitions, Biology: Ion channels, diffusion in cells

From the narrow escape problem to an eigenvalue problem

Key object: the QSD ν_ε

$$\nu_\varepsilon := \lim_{t \to +\infty} \mathrm{Law}(X_t \mid t < \tau_\varepsilon)$$

In the limit $\varepsilon \to 0$, X_t reaches ν_ε long before it escapes, so all statistics reduce to an **eigenvalue problem**

If $X_0 \sim \nu_\varepsilon$:

- $\tau_\varepsilon \sim \mathrm{Exp}(\lambda_\varepsilon)$ independent of X_{τ_ε}
- $X_{\tau_\varepsilon} \sim \partial_n \nu_\varepsilon \big|_{\Gamma_D^\varepsilon}$

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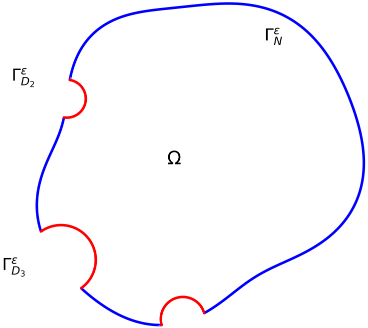
Eigenvalue problem for ν_ε

$$\begin{cases} -\Delta \nu_\varepsilon = \lambda_\varepsilon \nu_\varepsilon & \text{in } \Omega \\ \partial_n \nu_\varepsilon = 0 & \text{on } \Gamma_N^\varepsilon \\ \nu_\varepsilon = 0 & \text{on } \Gamma_D^\varepsilon \end{cases}$$



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Results on the exit time

Theorem [Asymptotic of the exit time]

Consider only one hole of radius r_ε. Then there exists a $C_{d,\Omega} > 0$ such that the eigenvalue λ_ε^0 scales as:

$$\lambda_\varepsilon^0 = \left(\mathbb{E}[\tau_\varepsilon]\right)^{-1} = \begin{cases} C_{d,\Omega}\, r_\varepsilon^{d-2} & + \mathrm{O}(r_\varepsilon^{d-1}), & \text{for } d > 3 \\ C_{3,\Omega}\, r_\varepsilon & + \mathrm{O}({r_\varepsilon}^2 \log(r_\varepsilon)), & \text{for } d = 3 \\ C_{2,\Omega}\, (\log(r_\varepsilon))^{-1} & + \mathrm{O}\left([\log(r_\varepsilon)]^{-2}\right), & \text{for } d = 2 \end{cases}$$

Where does the scaling come from? (Potential theory [1, 2])
The fundamental solution of the laplacian Λ in dimension d:

$$\lambda_\varepsilon^0 \sim C_{d,\Omega} \Lambda(r_\varepsilon)^{-1} \qquad \text{and} \qquad C_{d,\Omega} = \frac{\max\{d-2, 1\}}{2} \frac{|\mathscr{C}(0,1)|}{|\Omega|}$$

[1] Ammari et al. (2010), [2] Felli et al.(2020)

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