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The narrow escape problem

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PhD under the supervision of Tony Lelièvre, Urbain Vaes & Gabriel Stoltz

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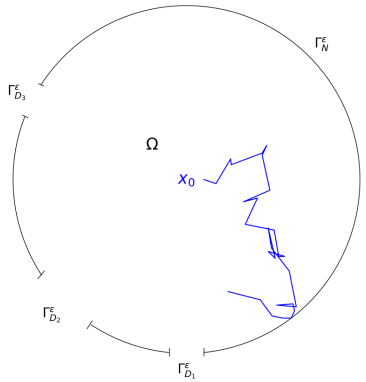
The narrow escape problem

Rare event: metastability induced by entropic barriers ($\neq$ energetic)



Setting:

- Domain Ω with holes Γ^{ε}_{D} and reflecting boundary Γ^{ε}_{N}
- A Brownian motion starting at x_0 taking a long time to exit $\tau_\varepsilon = \inf\{t \geq 0 \,|\, X_t \notin \overline{\Omega}\}$

$$\mathrm{d}X_t = \sqrt{2}\,\mathrm{d}B_t + \mathrm{d}L_t$$

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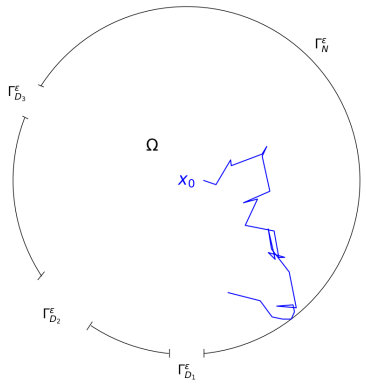

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Goal: In the limit of small holes $\varepsilon \to 0$:

- The law of the escape time τ_ε
- The law of exit hole X_{τ_ε}

Motivation: Entropic transitions, Biology: Ion channels, diffusion in cells

From the narrow escape problem to an eigenvalue problem

Key object:
the quasi-stationary distribution

$$\nu_\varepsilon = \lim_{t \to +\infty} \mathrm{Law}(X_t \mid t < \tau_\varepsilon)$$

In the limit $\varepsilon \to 0$,
X_t reaches ν_ε long before it escapes

If $X_0 \sim \nu_\varepsilon$:

- $\tau_\varepsilon \sim \mathrm{Exp}(\lambda_\varepsilon)$ independent of X_{τ_ε}
- $X_{\tau_\varepsilon} \sim \partial_n \nu_\varepsilon \big|_{\Gamma_D^\varepsilon}$

Learn more in: Di Gesù, Lelièvre, Le Peutrec and Nectoux (2016)

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The QSD ν_ε is the solution of

$$\begin{cases} -\Delta \nu_\varepsilon = \lambda_\varepsilon \nu_\varepsilon & \text{in } \Omega \\ \partial_n \nu_\varepsilon = 0 & \text{on } \Gamma_N^\varepsilon \\ \nu_\varepsilon = 0 & \text{on } \Gamma_D^\varepsilon \end{cases}$$

with $\lambda_\varepsilon > 0$ the smallest eigenvalue of the above problem

- Use of a quasi-mode ansatz to get the asymptotic expansion of ν_ε in the limit $\varepsilon \to 0$
- In any dimension $d \geq 2$, and smooth domains with N holes

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Results: which hole does the particle take?

Theorem [First order asymptotics]

N holes of radii $r_\varepsilon^{(i)} \to 0$: the rates of the holes add up

$$\lambda_\varepsilon \sim \sum_{i=1}^{N} \lambda_\varepsilon^{(i)}, \quad \lambda_\varepsilon^{(i)} \propto \begin{cases} |\log r_\varepsilon^{(i)}|^{-1} & d = 2 \\ \left(r_\varepsilon^{(i)}\right)^{d-2} & d \geq 3 \end{cases}$$

$$\mathbb{P}_{\nu_\varepsilon}\big(\text{exit through hole } i\big) \longrightarrow \frac{\lambda_\varepsilon^{(i)}}{\lambda_\varepsilon}$$

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Quiz: two holes, of radii ε and 2ε. Chance to exit through the small one?

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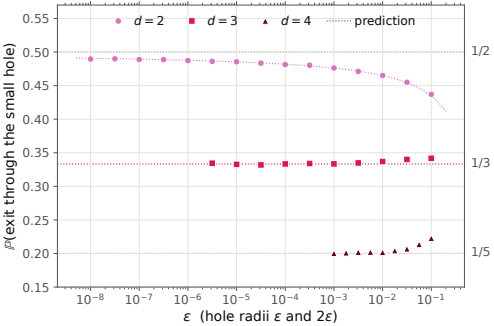
In $d = 2$, size does not matter!

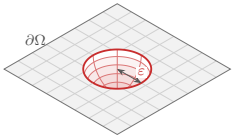


Walk-on-Stars simulations on the cube $[-1, 1]^d$

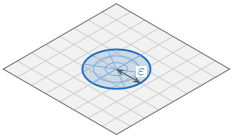
Quiz: two holes, of radii ε and 2ε. Chance to exit through the small one?

C., Lelièvre, Normand, Stoltz, Vaes, *The narrow escape problem in arbitrary dimension* (2026)

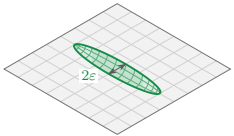
Recent development



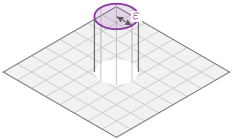
(A) Circular holes



(B) Flat holes



(C) Slits



(D) Tubes

New shapes of holes: the exit time is set by the geometry of the hole

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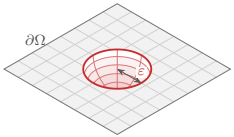

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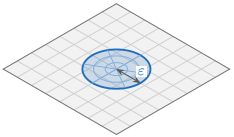

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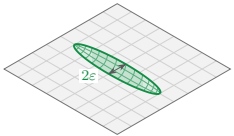

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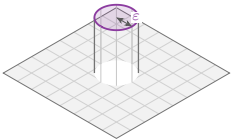

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With Boris Nectoux

Extend my work to more general diffusions:

$$\mathrm{d}X_t = b(X_t)\,\mathrm{d}t + \sigma(X_t)\,\mathrm{d}B_t + \mathrm{d}L_t$$

Next: narrow escape in fractals with Vincent and Loïs