Entropic metastability in the narrow escape problem

Louis Carillo

PhD under the supervision of Tony Lelièvre, Urbain Vaes & Gabriel Stoltz

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In collaboration with:

Urbain Vaes



Tony Lelièvre



Gabriel Stoltz



Thomas Normand



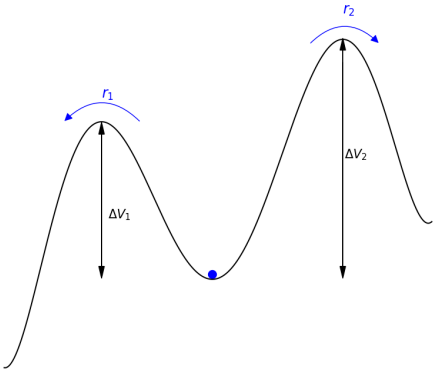
Loïs Delande



Boris Nectoux



Metastability of **energetic** origin



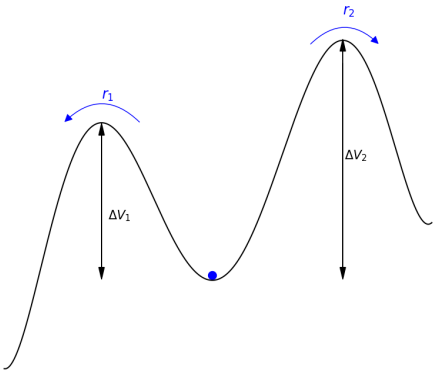
Thermal particle living in a potential well:

- Slow dynamics between the wells
- Long time to escape. This is a **rare event**

Toy model: Langevin particle in a double-well potential

How much time does it take to **escape** the well?

Metastability of **energetic** origin



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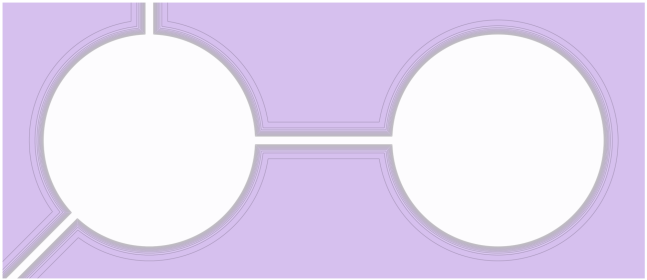
Eyring-Kramers' formula

The escape time is exponentially distributed, with a rate r_i, with $i \in \{1, 2\}$:

$$r_i = C_i \exp\left(-\frac{\Delta V_i}{k_B T}\right)$$

What if energy is not the driving factor?



A potential made of a confining well and a few **narrow canals**:

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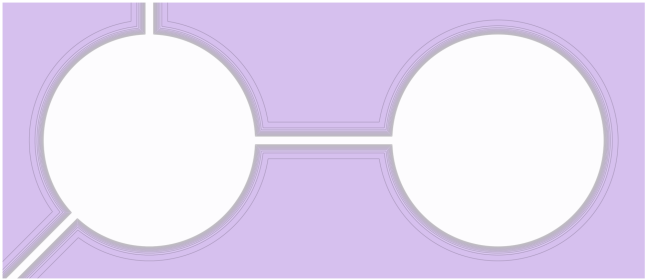

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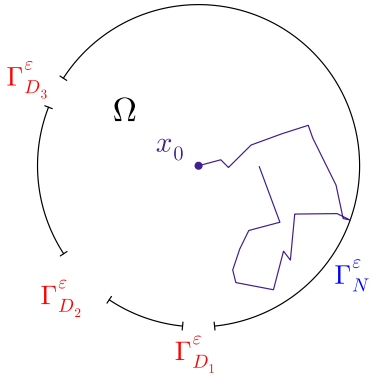
Is there an equivalent to the Eyring-Kramers formula in this case?

The narrow escape problem

Toy model of the metastability of entropic origin:



Setting:

- Domain Ω with holes Γ^{ε}_{D} and reflecting boundary Γ^{ε}_{N}
- A Brownian motion starting at x_0 taking a long time to exit $\tau_\varepsilon = \inf\{t \geq 0 \,|\, X_t \notin \overline{\Omega}\}$

$$\mathrm{d}X_t = \sqrt{2}\,\mathrm{d}B_t - \mathbb{1}_{\Gamma^{\varepsilon}_{N}}(X_t)n(X_t)\mathrm{d}L_t.$$

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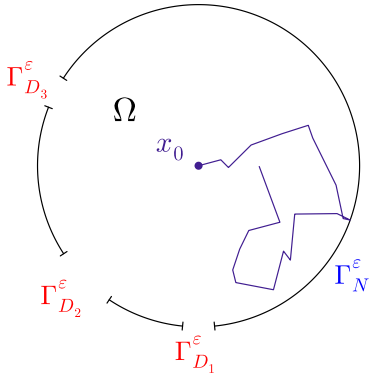

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Goal: In the limit of small holes $\varepsilon \to 0$:

- The law of the escape time τ_ε
- The law of exit hole X_{τ_ε}

References

Motivated by biology: ion channels, receptors on cell membranes, diffusion of a molecule in a cell...

In the physics literature for Ω a disk or a ball:

- Introduced by Holcman and Schuss in 2004
- A lot of formal results for $\mathbb{E}[\tau_\varepsilon]$: Bénichou, Voituriez 2008 Cheviakov, Kolokolnikov, Pierce, Pillay, Strauss, Ward 2010

In the math literature still in simple geometries

- Matched Asymptotics by Chen, Friedman 2011
- Layer potential techniques by Ammari, Kalimeris, Kang, Lee 2012
- Quasi-stationary approach for $\mathbb{E}[\tau_\varepsilon]$ and $\mathrm{Law}(X_{\tau_\varepsilon})$ by Lelièvre, Rachid, Stoltz 2024

A spectral approach to solve the narrow escape problem

Starting from $X_0 \sim \rho_0$, let $\rho(x,t)$ be the probability density of the particle to be still in Ω at time t and point x

$$\mathbb{P}\big(X_t \in A,\ t < \tau_\varepsilon\big) = \int_A \rho(x,t)\,\mathrm{d}x \qquad \text{so that} \qquad \mathbb{P}\big(\tau_\varepsilon > t\big) = \int_\Omega \rho(x,t)\,\mathrm{d}x$$

It verifies the Fokker–Planck equation

$$\begin{cases} \partial_t \rho = \Delta \rho & \text{in } \Omega, \qquad \partial_n \rho = 0 \quad \text{on } \Gamma_N^\varepsilon \text{ (reflecting)}, \\ \rho(\cdot,0) = \rho_0 & \text{in } \Omega, \qquad \rho = 0 \quad \text{on } \Gamma_D^\varepsilon \text{ (absorbing)}. \end{cases}$$

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Express ρ with the eigenvectors $(u_\varepsilon^k)_{k<m}$ of the matrix M, with eigenvalues $(\lambda_\varepsilon^k)_{k<m}$

$$-Mu_\varepsilon^k = \lambda_\varepsilon^k u_\varepsilon^k, \qquad 0 < \lambda_\varepsilon^0 < \lambda_\varepsilon^1 \leq \cdots \leq \lambda_\varepsilon^{m-1}$$

$$\rho(x,t) = \sum_{k=0}^{m-1} \langle \rho_0,\, u_\varepsilon^k \rangle \mathrm{e}^{-\lambda_\varepsilon^k t} u_\varepsilon^k(x)$$

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Express ρ with the eigenfunctions $(u_\varepsilon^k)_{k\geq 0}$ of the Laplacian Δ, with eigenvalues $(\lambda_\varepsilon^k)_{k\geq 0}$

$$-\Delta u_\varepsilon^k = \lambda_\varepsilon^k u_\varepsilon^k, \qquad 0 < \lambda_\varepsilon^0 < \lambda_\varepsilon^1 \leq \lambda_\varepsilon^2 \leq \dots$$

$$\rho(x,t) = \sum_{k\geq 0} \langle \rho_0,\, u_\varepsilon^k \rangle \mathrm{e}^{-\lambda_\varepsilon^k t} u_\varepsilon^k(x)$$

Large time behaviour

If the eigengap $\lambda_\varepsilon^1 - \lambda_\varepsilon^0$ is large, then the large time behaviour of $\rho(x,t)$ is dominated by the first eigenmode

$$\rho(x,t) = \sum_{k\geq 0} \langle \rho_0,\, u_\varepsilon^k \rangle \mathrm{e}^{-\lambda_\varepsilon^k t} u_\varepsilon^k(x) \approx \langle \rho_0,\, u_\varepsilon^0 \rangle \mathrm{e}^{-\lambda_\varepsilon^0 t} u_\varepsilon^0(x)$$

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From the probability, we deduce the behaviour of quantities of interest:

- The exit time distribution is exponentially distributed with parameter λ_ε^0
- The law of the exit hole will be dictated by u_ε^0
- Starting from $\rho_0 \propto u_\varepsilon^0$, these results are exact

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$\Rightarrow$ Rigorous approach: the quasi-stationary distribution [1]

[1] Di Gesù, Lelièvre, Le Peutrec and Nectoux, *Faraday Discussion*, (2016)

Quasi-stationary distribution (QSD)

Definition

A quasi-stationary distribution ν_ε is a probability measure on $\overline{\Omega}$ such that

$$\text{If } X_0 \sim \nu_\varepsilon, \quad \text{then } \mathrm{Law}(X_t \mid t < \tau_\varepsilon) = \nu_\varepsilon$$

Starting from $\rho_0 = u_\varepsilon^0$, for any $x \in \Omega$

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Consequences:

- The quasi-stationary distribution is the first eigenmode up to a renormalisation
- The QSD is the Yaglom limit of the process

$$\text{If } X_0 \sim x_0 \in \Omega, \quad \text{then } \lim_{t \to +\infty} \operatorname{Law}(X_t \mid t < \tau_\varepsilon) = \nu_\varepsilon$$

First result on the QSD

Theorem [C., Lelièvre, Normand, Stoltz, Vaes]

There exists a unique quasi-stationary distribution ν_ε associated to the process $(X_t)_{t\geq 0}$.
It can be written as $\nu_\varepsilon(\mathrm{d}x) = u_\varepsilon^0(x)\,\mathrm{d}x$

Corollary

Starting from $X_0 \sim \nu_\varepsilon$, we know

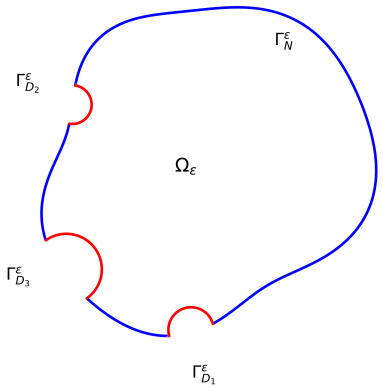
- The exit time $\tau_\varepsilon \sim \mathrm{Exp}(\lambda_\varepsilon^0)$
- The exit place $\mathbb{P}(X_{\tau_\varepsilon} \in \Gamma_k^\varepsilon) = -\frac{1}{\lambda_\varepsilon^0}\partial_n \nu_\varepsilon(\Gamma_k^\varepsilon)$

Furthermore, τ_ε and X_{τ_ε} are independent

Theorem [C., Delande, Nectoux]

These results also hold for diffusions of the form $\mathrm{d}X_t = b(X_t)\mathrm{d}t + \sigma(X_t)\mathrm{d}W_t + n(X_t)\mathrm{d}L_t$

The QSD as an eigenvalue problem



We want to find the QSD ν_ε :

$$\begin{cases} -\Delta \nu_\varepsilon = \lambda_\varepsilon \nu_\varepsilon & \text{in } \Omega_\varepsilon \\ \partial_n \nu_\varepsilon = 0 & \text{on } \Gamma_N^\varepsilon \\ \nu_\varepsilon = 0 & \text{on } \Gamma_D^\varepsilon \end{cases}$$

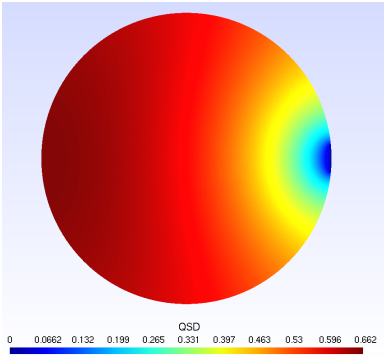
N holes of radius $r_\varepsilon^{(i)}$ centered at $x^{(i)} \in \partial\Omega$

Already studied in [2] for the disk and the ball

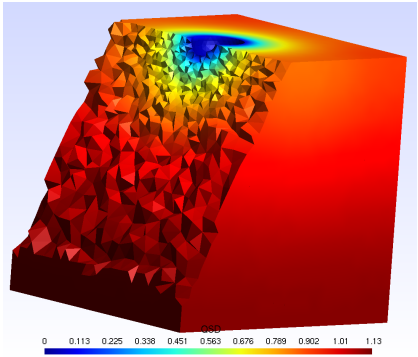
Use of a quasi-mode ansatz to get the asymptotic expansion of ν_ε in the limit $\varepsilon \to 0$

[2] Lelièvre, Rachid and Stoltz, *preprint* (2024)

What does the QSD look like?



Dimension 2: disk with one hole



Dimension 3: cube with one hole

- ν_ε is almost flat: the particle equilibrates long before it escapes
- It only vanishes in a small layer around the hole

Results on the exit time

Theorem [Asymptotic of the exit time]

Consider only one hole of radius r_ε. Then there exists a $C_{d,\Omega} > 0$ such that the eigenvalue λ_ε scales as:

$$\lambda_\varepsilon = \Big(\mathbb{E}[\tau_\varepsilon]\Big)^{-1} \underset{\varepsilon \to 0}{\sim} \begin{cases} \dfrac{C_{2,\Omega}}{|\log(r_\varepsilon)|} & \text{for } d = 2 \\ C_{d,\Omega}\, r_\varepsilon^{d-2} & \text{for } d \geq 3 \end{cases}$$

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Where does the scaling come from? The Coulomb potential Λ in dimension d:

$$\Lambda(r) = \begin{cases} \log(r) & \text{for } d = 2 \\ -\, r^{2-d} & \text{for } d \geq 3 \end{cases} \qquad \Longrightarrow \qquad \lambda_\varepsilon \sim -C_{d,\Omega} \Lambda(r_\varepsilon)^{-1}$$

$$\text{with} \quad C_{d,\Omega} = \frac{\max\{d-2,\, 1\}}{2} \frac{|\mathscr{C}(0,1)|}{|\Omega|}$$

Theorem [Asymptotic of the exit time]

Consider N holes of radii $r_\varepsilon^{(k)}$. Then there exists a $C_{d,\Omega} > 0$ such that the eigenvalue λ_ε scales as:

$$\lambda_\varepsilon = \Big(\mathbb{E}[\tau_\varepsilon]\Big)^{-1} \underset{\varepsilon \to 0}{\sim} \begin{cases} C_{2,\Omega} \displaystyle\sum_{k=1}^{N} \frac{1}{|\log(r_\varepsilon^{(k)})|} & \text{for } d = 2 \\ C_{d,\Omega} \displaystyle\sum_{k=1}^{N} \big(r_\varepsilon^{(k)}\big)^{d-2} & \text{for } d \geq 3 \end{cases}$$

Exit hole distribution for N holes

Theorem [Exit hole distribution]

The probability to exit through hole $i \in \{1, \dots, N\}$ is given by:

$$\mathbb{P}_{\nu_\varepsilon}(X_\tau \in \Gamma^\varepsilon_{D_i}) \underset{\varepsilon \to 0}{\sim} \frac{\lambda^i_\varepsilon}{\lambda_\varepsilon}$$

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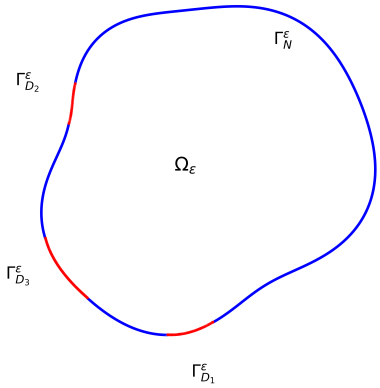
Example: with $N = 2$, scaling as $r^{(1)}_\varepsilon = \varepsilon$ and $r^{(2)}_\varepsilon = 2\varepsilon$

- in dimension 2, $\mathbb{P}_{\nu_\varepsilon}(X_\tau \in \Gamma^\varepsilon_{D_1}) \approx \frac{1}{2}$ equal to $\mathbb{P}_{\nu_\varepsilon}(X_\tau \in \Gamma^\varepsilon_{D_2})$
- in dimension 3, $\mathbb{P}_{\nu_\varepsilon}(X_\tau \in \Gamma^\varepsilon_{D_1}) \approx \frac{1}{3}$ twice smaller than $\mathbb{P}_{\nu_\varepsilon}(X_\tau \in \Gamma^\varepsilon_{D_2})$
- in dimension 4, $\mathbb{P}_{\nu_\varepsilon}(X_\tau \in \Gamma^\varepsilon_{D_1}) \approx \frac{1}{5}$ four times smaller than $\mathbb{P}_{\nu_\varepsilon}(X_\tau \in \Gamma^\varepsilon_{D_2})$

What about flat holes?



Take holes that follow the curvature of Ω,

that is a much harder problem

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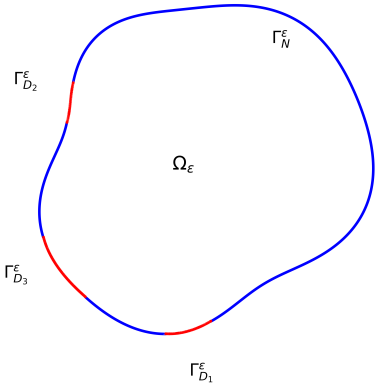

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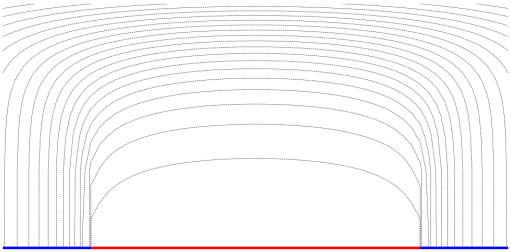
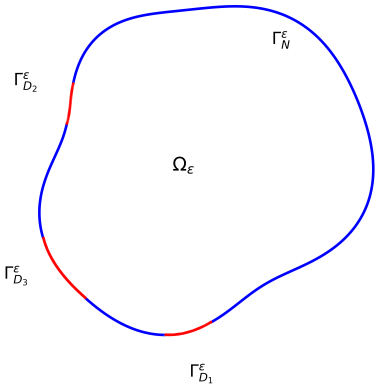


Level curves of the solution ν_ε near a flat hole

What about flat holes?



Theorem [Exit time for flat holes]

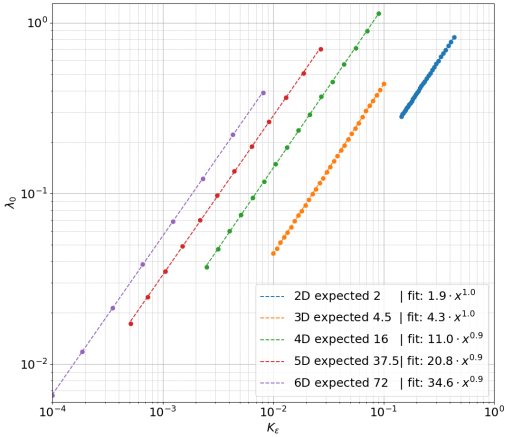
In this setting:

$$\lambda_\varepsilon^{\text{flat}} \underset{\varepsilon\to 0}{\sim} \begin{cases} \dfrac{C_{2,\Omega}^{\text{flat}}}{|\log(r_\varepsilon)|} & \text{for } d=2 \\ C_{d,\Omega}^{\text{flat}}\, r_\varepsilon^{d-2} & \text{for } d\geq 3 \end{cases}$$

Comparison with circular holes:

- $d=2$: $C_{2,\Omega}^{\text{flat}} = C_{2,\Omega}$ (Brownian motion is recurrent in dimension 2)
- $d=3$: $C_{3,\Omega}^{\text{flat}} = \dfrac{2}{\pi}\, C_{3,\Omega}$

Why naive Monte Carlo fails



Ball in dimension d, theory vs simulation:

d	C_d^{ball}	C_d^{ball} (simu)
2	2	1.9
3	4.5	4.3
4	16	11
5	37.5	20.8
6	72	34.6

1 week on the Cermics cluster and still wrong for $d \geq 4$

Why naive Monte Carlo fails

Euler–Maruyama: $X_{n+1} = X_n + \sqrt{2\Delta t}\,\xi_n$, reflected on Γ_N^ε, stopped on Γ_D^ε

- The time step must resolve the hole: $\Delta t \ll \varepsilon^2$
- The exit time diverges: $\mathbb{E}[\tau_\varepsilon] \propto \varepsilon^{-(d-2)}$

$\rightsquigarrow$ cost per trajectory $\gtrsim \dfrac{\mathbb{E}[\tau_\varepsilon]}{\varepsilon^2} \propto \varepsilon^{-d}$

Example: $d = 3$, $\varepsilon = 10^{-6}$: 10^{18} steps per trajectory

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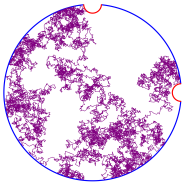
A more efficient simulation method: walk-on methods

Key idea: the Brownian motion is predictable on average

$\Rightarrow$ cheaper to use these averages than to simulate every small step

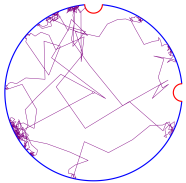
- **Walk on Spheres**: jump onto the largest sphere avoiding the boundary
  [Muller, *Ann. Math. Statist.* (1956)]
- **Walk on Stars**: jump onto the largest star-shaped region, exact reflection on Γ_N^ε
  [Sawhney, Miller, Gkioulekas, Crane, *ACM Trans. Graph.* (2023)]

Euler–Maruyama



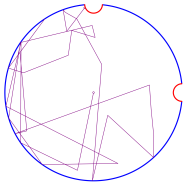
$\approx$ 8500 steps

Walk on Spheres



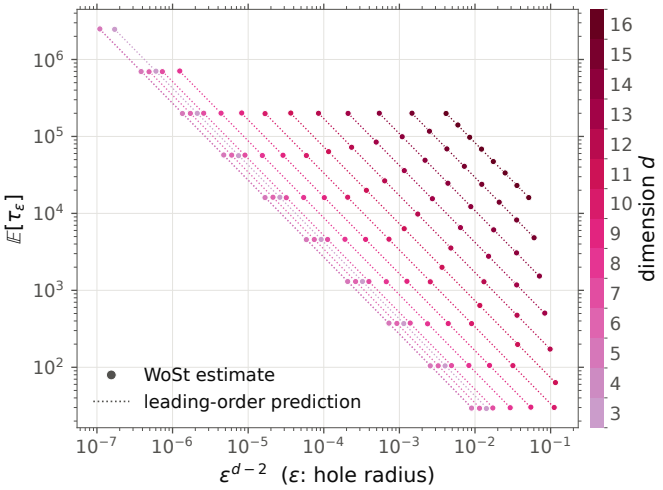
$\approx$ 350 steps

Walk on Stars



$\approx$ 15 jumps

Walk on Stars: exit time



WoSt simulations with 1 hole on the d-dimensional cube $[-1, 1]^d$

We fit the law

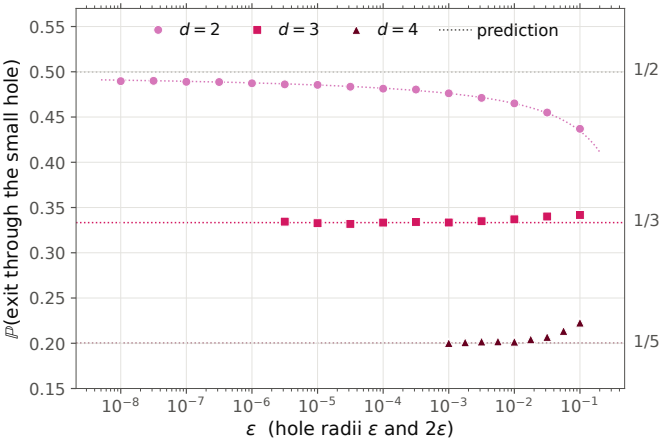
$$\mathbb{E}[\tau_\varepsilon] = C_d\, \varepsilon^{-\alpha}$$

For every $3 \leq d \leq 16$:

- $\alpha = d - 2$ within ± 0.05
- C_d matches the theory within 0.4%

A night of computation on the Cermics cluster

Walk on Stars: exit hole

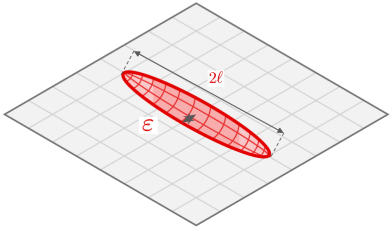


WoSt simulations on the d-dimensional cube $[-1, 1]^d$, with 1 hole of radius ε and 1 hole of radius 2ε

Plot $\mathbb{P}$(exit by the small hole)

- In agreement with the theory
- In $d = 2$, the convergence is slow, in $1/|\log(\varepsilon)|$

What about slits?



Slit of length 2ℓ and half-width ε on $\partial\Omega$

The slit reduces to a segment, not a point

Theorem [Exit time for slits]

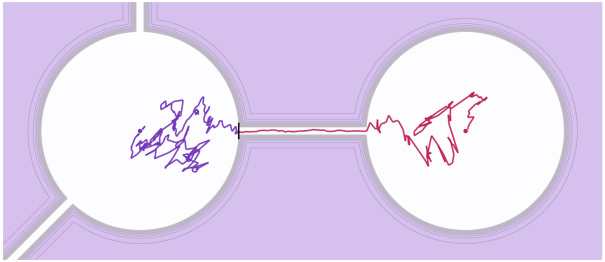
In dimension $d = 3$, with one slit:

$$\lambda_\varepsilon^{\text{slit}} \underset{\varepsilon \to 0}{\sim} \frac{2\pi}{|\Omega|} \frac{\ell}{|\log(\varepsilon)|}$$

Comparison with point-like holes:

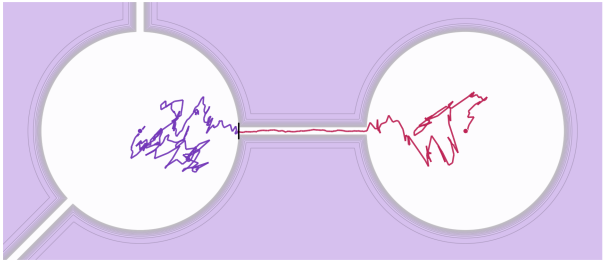
- Same $|\log(\varepsilon)|$ scaling as a hole in $d = 2$: what matters is the codimension of the hole
- In dimension d, we expect $\lambda_\varepsilon^{\text{slit}}(d) \propto \lambda_\varepsilon(d-1)$

Back to the narrow canals: the narrow transition



Unlike energetic barriers, the transition time $\Omega \to \Omega'$ is not twice the escape time

Back to the narrow canals: the narrow transition



Unlike energetic barriers, the transition time $\Omega \to \Omega'$ is not twice the escape time

Theorem [Exit through tubes]

In $d = 3$, tubes of length L_k and section $A_\varepsilon^{(k)} \to 0$:

$$\mathbb{E}_{\nu_\varepsilon}[\tau_\varepsilon] \underset{\varepsilon \to 0}{\sim} |\Omega| \left(\sum_k \frac{A_\varepsilon^{(k)}}{L_k} \right)^{-1}, \qquad \mathbb{P}_{\nu_\varepsilon}(\text{tube } k) \underset{\varepsilon \to 0}{\sim} \frac{A_\varepsilon^{(k)}/L_k}{\sum_j A_\varepsilon^{(j)}/L_j}$$

An entropic Eyring–Kramers law

Energetic barrier (Eyring–Kramers), minimum x, saddle point z:

$$\mathbb{E}[\tau] \sim \frac{2\pi}{|\lambda_1(z)|} \sqrt{\frac{|\det \nabla^2 V(z)|}{\det \nabla^2 V(x)}} \exp\left(\frac{V(z) - V(x)}{k_{\mathrm{B}} T}\right)$$

Narrow transition: one tube of length L and section A_ε, $\quad \mathbb{E}[\tau] \simeq |\Omega|\, L\, A_\varepsilon^{-1}$

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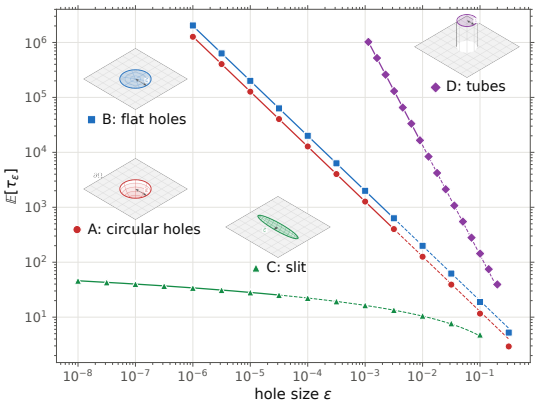
	Energetic	Entropic
Time to cross the saddle / tube	$\dfrac{2\pi}{\lvert\lambda_1(z)\rvert}$	L^2
Volume of the well / transition region	$\sqrt{\dfrac{\lvert\det \nabla^2 V(z)\rvert}{\det \nabla^2 V(x)}}$	$\dfrac{\lvert\Omega\rvert}{A_\varepsilon L}$
Energy barrier	$\exp\left(\dfrac{\Delta V}{k_{\mathrm{B}}T}\right)$	1

Recap



Exit time in dimension $d = 3$, as the target shrinks ($\varepsilon \to 0$):

Target	ε	$\mathbb{E}[\tau_\varepsilon]$
A: circular hole	radius	ε^{-1}
B: flat hole	radius	ε^{-1}
C: slit	half-width	$\lvert\log(\varepsilon)\rvert$
D: tube	tube radius	ε^{-2}

The scaling is set by the geometry of the target, not by an energy barrier

Conclusion

- The narrow escape is a toy model of metastability of entropic origin
- With our approach we can solve it for any (locally) smooth domain in any dimension
- We get the asymptotic expansion of the exit time and the law of exit hole
- Narrow transition: an entropic Eyring–Kramers law

Future work:

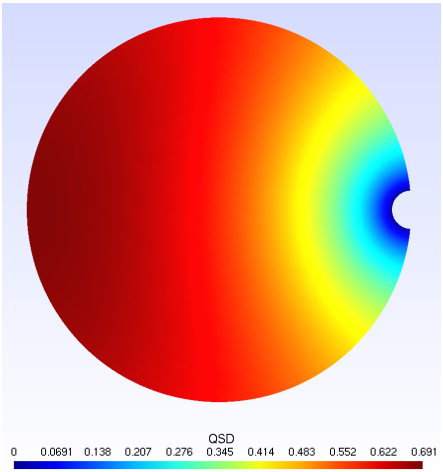
- Could we do the same for active particles ?

My work on this topic:

- C., Lelièvre, Normand, Vaes, *A spectral approach to the narrow escape problem in two-dimensional domains*, arXiv:2608.05007
- C., Lelièvre, Normand, Stoltz, Vaes, *The narrow escape problem in arbitrary dimension*, arXiv:2608.02212
- C., *Narrow escape problems beyond spherical holes: slits and tubes*, arXiv:2610.26514
- C., Delande, Nectoux, *Entropic metastability for drifted reflected Brownian motion: quasi-stationary distributions, regularity, and the exit position distribution*, arXiv:2610.26378

How to build the quasimode (1 hole)?

From the simulations, u_ε^0 is almost constant far from the hole:



We can approximate the solution u_ε^0 by a quasimode (semi-classical technique):

$$u_0^\varepsilon \simeq \varphi_\varepsilon = 1 + K_\varepsilon f$$

with K_ε the approximation of the eigenvalue and f the solution when the hole is a point:

$$\begin{cases} -\Delta f = 1 & \text{in } \Omega \\ \partial_n f = 0 & \text{on } \partial\Omega \backslash \{x^{(h)}\} \end{cases}$$

with $x^{(h)}$ the center of the hole.

Construction of the quasimode

By the compatibility equation

$$|\Omega| = \int_\Omega \Delta f = \int_{\partial\Omega} \partial_n f$$

But $\partial_n f = 0$ on $\partial\Omega \backslash \{x^{(h)}\}$, so $\partial_n f$ is not a function but a distribution. A part of my PhD is to have approximation of the singularity of the solution of in the weak $W^{1,p}(\Omega)$ sense:

$$\begin{cases} -\Delta f = 1 & \text{in } \Omega \\ \partial_n f = -|\Omega| \delta_{x^{(h)}} & \text{on } \partial\Omega \end{cases}$$

with $f \in \left\{ u \in W^{1,p}(\Omega),\ \Delta u = 1 \in L^p(\Omega), \partial_n u = -|\Omega|\delta_{x^{(h)}} \in W^{-\frac{1}{p},p}(\Omega) \right\}$

Construction of the quasimode

Explicit expressions for simple geometries [2], ex: the unit sphere with one hole on $x^{(h)}$

$$\varphi_\varepsilon(x) = 1 + \frac{1}{\sqrt{3\pi}} K_\varepsilon^{(h)} \left[\frac{1}{|x - x^{(h)}|} - \frac{1}{2} \log \left(1 - x \cdot x^{(h)} + \left| x - x^{(h)} \right| \right) + \frac{|x|^2}{4} \right]$$

But in the general case, we have been able to do expansions around the singularity only in the following settings:

- **dimension** 2, any smooth domain for flat holes
- **dimension** 2 **and above**, any C^2 domain, locally smooth, with circular holes
- **dimension** 3, any smooth domain for slit-like holes

I will focus in the dimension 3 setting with N circular holes

Properties of the quasimode

Theorem [Quasimode properties]

One can construct a quasimode $\varphi_\varepsilon \in H^1(\Omega_\varepsilon)$ that satisfies:

- An approximate eigenvalue equation: with $C_{3,\Omega} \in \mathbb{R}^+$

$$\| -\Delta\varphi_\varepsilon - C_{3,\Omega}\overline{r_\varepsilon}\varphi_\varepsilon \|_{L^2(\widetilde{\Omega}_\varepsilon)} = \mathrm{O}\left((\overline{r_\varepsilon})^2\right)$$

- The Neumann condition:

$$\partial_n \varphi_\varepsilon = 0 \quad \text{on } \Gamma_N^\varepsilon$$

- An approximate Dirichlet condition:

$$\|\varphi_\varepsilon\|_{L^\infty(\widetilde{\Gamma}_D^\varepsilon)} = \mathrm{O}\left(\overline{r_\varepsilon}\log(\overline{r_\varepsilon})\right)$$

where $\overline{r_\varepsilon} = r_\varepsilon^{(1)} + \cdots + r_\varepsilon^{(N)}$

From the quasimode to the eigenvalue

With Green identity

$$\lambda_\varepsilon = \frac{\langle \varphi_\varepsilon,\, -\Delta u_0^\varepsilon \rangle_{\widetilde{\Omega}_\varepsilon}}{\langle \varphi_\varepsilon,\, u_0^\varepsilon \rangle_{\widetilde{\Omega}_\varepsilon}} = \frac{\langle -\Delta \varphi_\varepsilon,\, u_0^\varepsilon \rangle_{\widetilde{\Omega}_\varepsilon} + \langle \partial_n \varphi_\varepsilon,\, u_0^\varepsilon \rangle_{\partial\widetilde{\Omega}_\varepsilon} - \langle \varphi_\varepsilon,\, \partial_n u_0^\varepsilon \rangle_{\partial\widetilde{\Omega}_\varepsilon}}{\langle \varphi_\varepsilon,\, u_0^\varepsilon \rangle_{\widetilde{\Omega}_\varepsilon}}$$

$$= C_{3,\Omega}\overline{r_\varepsilon} + \mathrm{O}\left((\overline{r_\varepsilon})^2\right) - \frac{\langle \varphi_\varepsilon,\, \partial_n u_0^\varepsilon \rangle_{\widetilde{\Gamma}_D^\varepsilon}}{\langle \varphi_\varepsilon,\, u_0^\varepsilon \rangle_{\widetilde{\Omega}_\varepsilon}}$$

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$$= C_{3,\Omega}\overline{r_\varepsilon} + \mathrm{O}\left((\overline{r_\varepsilon})^2\right) - \frac{\langle \varphi_\varepsilon,\, \partial_n u_0^\varepsilon \rangle_{\widetilde{\Gamma}_D^\varepsilon}}{\langle \varphi_\varepsilon,\, u_0^\varepsilon \rangle_{\widetilde{\Omega}_\varepsilon}}$$

Since $\varphi_\varepsilon = 1 + \mathrm{O}_{L^2(\widetilde{\Omega}_\varepsilon)}(\overline{r_\varepsilon})$ and $\varphi_\varepsilon = \mathrm{O}_{L^\infty(\widetilde{\Gamma}_D^\varepsilon)}(\overline{r_\varepsilon}\log(\overline{r_\varepsilon}))$

$$\frac{\langle \varphi_\varepsilon,\, \partial_n u_0^\varepsilon \rangle_{\widetilde{\Gamma}_D^\varepsilon}}{\langle \varphi_\varepsilon,\, u_0^\varepsilon \rangle_{\widetilde{\Omega}_\varepsilon}} = \frac{\langle 1,\, \partial_n u_0^\varepsilon \rangle_{\widetilde{\Gamma}_D^\varepsilon}}{\int_{\widetilde{\Omega}_\varepsilon} u_0^\varepsilon}\, \mathrm{O}(\overline{r_\varepsilon}\log(\overline{r_\varepsilon})) = \lambda_0^\varepsilon\, \mathrm{O}(\overline{r_\varepsilon}\log(\overline{r_\varepsilon}))$$

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Then

$$\lambda_0^\varepsilon = C_{3,\Omega}\overline{r_\varepsilon} + \mathrm{O}\left((\overline{r_\varepsilon})^2 \log(\overline{r_\varepsilon})\right)$$

Proof of the result on the exit hole distribution (back in dimension 3)

Consider the quasimode $\varphi_\varepsilon^{(-j)}$ without the hole j, it verifies on $\Gamma_{\mathrm{D}}^\varepsilon$

$$\varphi_\varepsilon^{(-j)} = \mathbb{1}_{\Gamma_{\mathrm{D}_j}^\varepsilon} + \mathrm{O}_{L^\infty}(\overline{K_\varepsilon}\log(\overline{K_\varepsilon}))$$

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Then the probability can be writen as

$$\mathbb{P}_{\nu_\varepsilon}(X_\tau \in \Gamma_{D_j}^\varepsilon) = -\frac{\langle \partial_n u_0^\varepsilon, \mathbb{1}_{\Gamma_{\mathrm{D}_i}^\varepsilon}\rangle_{\Gamma_D^\varepsilon}}{\langle \partial_n u_0^\varepsilon, 1\rangle_{\Gamma_D^\varepsilon}} = -\frac{\langle \partial_n u_0^\varepsilon, \varphi_\varepsilon^{(-j)}\rangle_{\Gamma_D^\varepsilon}}{\langle \partial_n u_0^\varepsilon, 1\rangle_{\Gamma_D^\varepsilon}} + \mathrm{O}(\overline{K_\varepsilon}\log(\overline{K_\varepsilon}))$$

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Using the Green identity,

$$\begin{aligned}\mathbb{P}_{\nu_\varepsilon}(X_\tau \in \Gamma_{D_j}^\varepsilon) &= -\frac{\langle \Delta u_0^\varepsilon, \varphi_\varepsilon^{(-j)}\rangle_\Omega - \langle u_0^\varepsilon, \Delta\varphi_\varepsilon^{(-j)}\rangle_\Omega}{\langle \partial_n u_0^\varepsilon, 1\rangle_{\Gamma_D^\varepsilon}} + \mathrm{O}(\overline{K_\varepsilon}\log(\overline{K_\varepsilon})) \\ &= \frac{C_{3,\Omega}}{\lambda_0^\varepsilon} K_\varepsilon^j + \mathrm{O}(\overline{K_\varepsilon}\log(\overline{K_\varepsilon}))\end{aligned}$$