Entropic metastability in the narrow escape problem

Louis Carillo

PhD under the supervision of Tony Lelièvre, Urbain Vaes & Gabriel Stoltz

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In collaboration with:

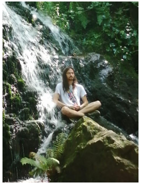
Urbain Vaes

Tony Lelièvre

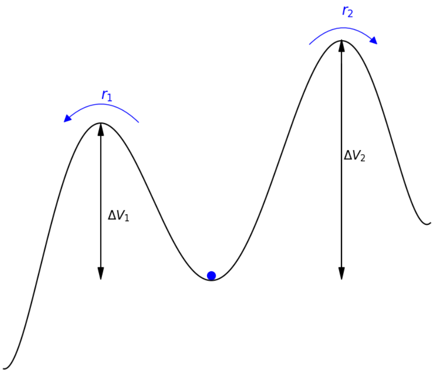
Thomas Normand

Gabriel Stoltz

Loïs Delande



Metastability of **energetic** origin



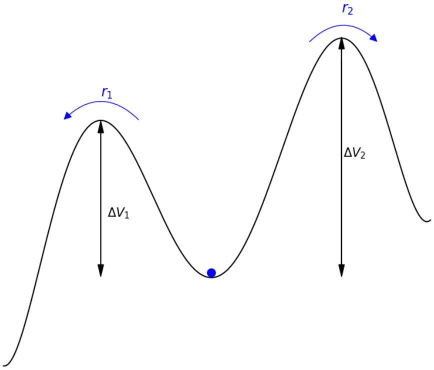
Thermal particle living in a potential well:

- Slow dynamics between the wells
- Long time to escape. This is a **rare event**

Toy model: Langevin particle in a double-well potential

How much time does it take to **escape** the well?

Metastability of **energetic** origin



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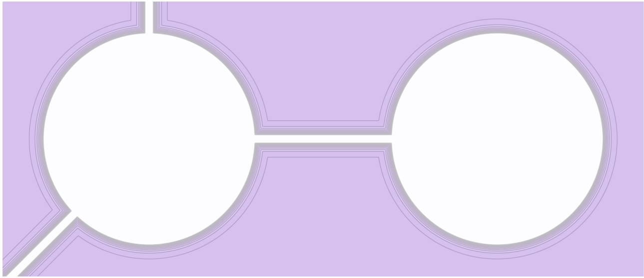
Eyring-Kramers' formula

The escape time is exponentially distributed, with a rate r_i, with $i \in \{1, 2\}$:

$$r_i = C_i \exp\left(-\frac{\Delta V_i}{k_\mathrm{B} T}\right)$$

What if energy is not the driving factor?

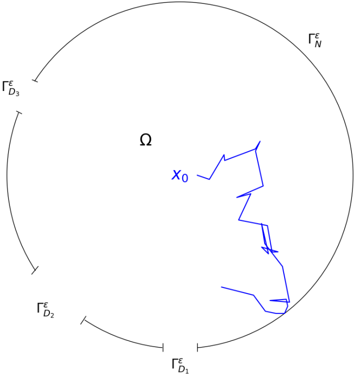


A potential made of a confining well and a few **narrow canals**:

Still a long time to escape. This is still a **rare event**

Is there an equivalent to the Eyring-Kramers formula in this case?

Toy model of the metastability of entropic origin:

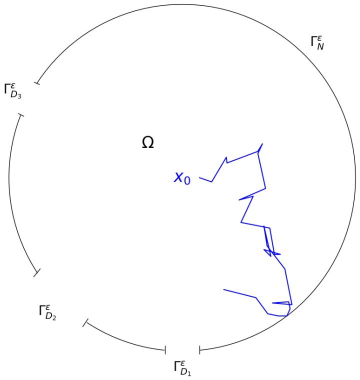


Setting:

- Domain Ω with holes Γ_D^ε and reflecting boundary Γ_N^ε
- A Brownian motion starting at x_0 taking a long time to exit $\tau_\varepsilon = \inf\{t \geq 0 \,|\, X_t \notin \overline{\Gamma}_D^\varepsilon\}$

$$\mathrm{d}X_t = \sqrt{2}\,\mathrm{d}B_t - \mathbb{1}_{\partial\Omega}(X_t)n(X_t)\mathrm{d}L_t.$$

Toy model of the metastability of entropic origin:



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Goal: In the limit of small holes $\varepsilon \to 0$:

- The law of the escape time τ_ε
- The law of exit hole X_{τ_ε}

References

Motivated by biology: ion channels, receptors on cell membranes, diffusion of a molecule in a cell...

In the physics literature for Ω a disk or a ball:

- Introduced by Holcman and Schuss in 2004
- A lot of formal results for $\mathbb{E}[\tau_\varepsilon]$: Bénichou, Voituriez 2008 Cheviakov Kolokolnikov, Pierce, Pillay, Straube, Ward 2010

In the math literature still in simple geometries

- Matched Asymptotics by Chen, Friedman 2011
- Layer potential techniques by Ammari, Kalimeris, Kang, Lee 2012
- Quasi-stationary approach for $\mathbb{E}[\tau_\varepsilon]$ and $\mathrm{Law}(X_{\tau_\varepsilon})$ by Lelièvre, Rachid, Stoltz 2024

In all literature, there is a difficulty to have precise results in the vicinity of the holes

Let $\rho(x,t)$ be the unormalised Law of X_t conditioned on the fact it has not yet escaped starting from ρ_0 . It verifies the Fokker-Plank equation:

$$\begin{cases} \partial_t \rho = \Delta \rho & \text{in } \Omega \\ \partial_n \rho = 0 & \text{on } \Gamma_N^\varepsilon \\ \rho = 0 & \text{on } \Gamma_D^\varepsilon \\ \rho(\cdot, 0) = \rho_0 & \text{in } \Omega \end{cases} \tag{1}$$

Idea of the proof: Itô's lemma, the local time is continuous with finite variation

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The operator $\mathcal{L}_\varepsilon$ associated to (1) is self-adjoint with compact resolvent

→ there exists an orthonormal basis in $H^1(\Omega)$ of eigenfunctions $(u_\varepsilon^k)_{k\geq 0}$ and eigenvalues $(\lambda_\varepsilon^k)_{k\geq 0}$ ranked in increasing order $\lambda_\varepsilon^k < \lambda_\varepsilon^1 < \dots$

$$\rho(x,t) = \sum_{k\geq 0} \langle \rho_0,\, u_\varepsilon^k \rangle \mathrm{e}^{-\lambda_\varepsilon^k t} u_\varepsilon^k(x)$$

Large time behaviour

If the eigengap $\lambda_\varepsilon^1 - \lambda_\varepsilon^0$ is large, then the large time behaviour of $\rho(x,t)$ is dominated by the first eigenmode

$$\rho(x,t) = \sum_{k\geq 0} \langle \rho_0,\, u_\varepsilon^k \rangle \mathrm{e}^{-\lambda_\varepsilon^k t} u_\varepsilon^k(x) \approx \langle \rho_0,\, u_\varepsilon^0 \rangle \mathrm{e}^{-\lambda_\varepsilon^0 t} u_\varepsilon^0(x) \tag{2}$$

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From the probability, we deduce the behaviour of quantities of interest:

- The exit time distribution is exponentially distributed with parameter λ_ε^0
- The law of the exit hole will be dictated by u_ε^0
- If we start from $\rho_0 = u_\varepsilon^0$, (2) is exact

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$\Rightarrow$ Rigorous approach: the quasi-stationary distribution [1]

[1] Di Gesù, Lelièvre, Le Peutrec and Nectoux, *Faraday Discussion*, (2016)

Quasi-stationary distribution (QSD)

Definition by the Yaglom limit

$$\text{If } X_0 \in \Omega, \quad \text{then} \quad \lim_{t \to +\infty} \mathrm{Law}(X_t \,|\, t < \tau_\varepsilon) = \nu_\varepsilon$$

Before, we had, starting from $\rho_0 = u_\varepsilon^0$, for any $x \in \Omega$

$$\rho(x,t) \propto \mathrm{e}^{-\lambda_\varepsilon t} u_\varepsilon^0(x)$$

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Consequences:

- The quasi-stationary distribution is the first eigenmode up to a renormalisation
- If $\rho_0 = \nu_\varepsilon$, then $\rho(\cdot, t) = \nu_\varepsilon$ up to a renormalisation for all $t \geq 0$
  the quasi-stationary distribution is stationary until the escape event

Fundamental properties of the QSD

Assume that $X_0 \sim \nu_\varepsilon$. Then

- The exit time τ_ε is *exponentially distributed* $\sim \mathrm{Exp}(\lambda_\varepsilon^\varepsilon)$

$$\begin{aligned}\mathbb{P}_{\nu_\varepsilon}[\tau_\varepsilon \geq s+t] &= \mathbb{P}_{\nu_\varepsilon}[\tau_\varepsilon \geq s+t \,|\, \tau_\varepsilon \geq s]\,\mathbb{P}_{\nu_\varepsilon}[\tau_\varepsilon \geq s] \\ &= \mathbb{P}_{\nu_\varepsilon}[\tau_\varepsilon \geq t]\,\mathbb{P}_{\nu_\varepsilon}[\tau_\varepsilon \geq s]\end{aligned}$$

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- The exit point X_{τ_ε} is *independent* of the exit time τ_ε

$$\begin{aligned}\mathbb{P}_{\nu_\varepsilon}[X_{\tau_\varepsilon} \in A, \tau_\varepsilon \geq t] &= \mathbb{P}_{\nu_\varepsilon}[X_{\tau_\varepsilon} \in A \mid \tau_\varepsilon \geq t]\,\mathbb{P}_{\nu_\varepsilon}[\tau_\varepsilon \geq t] \\ &= \mathbb{P}_{\nu_\varepsilon}[X_{\tau_\varepsilon} \in A]\,\mathbb{P}_{\nu_\varepsilon}[\tau_\varepsilon \geq t]\end{aligned}$$

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- The *law of the exit point* is given by

$$\mathbb{P}_{\nu_\varepsilon}(X_{\tau_\varepsilon} \in \Gamma^\varepsilon_{\mathrm{D}_i}) \propto \int_{\Gamma^\varepsilon_{\mathrm{D}_i}} \mathrm{d}\,(\partial_n \nu_\varepsilon)$$

First result on the QSD

Theorem[C., Lelièvre, Normand Stoltz, Vaes]

There exists a unique quasi-stationary distribution ν_ε associated to the process $(X_t)_{t\geq 0}$. It is the L^1-normalised principal eigenfunction of $-\Delta$ with mixed boundary conditions:

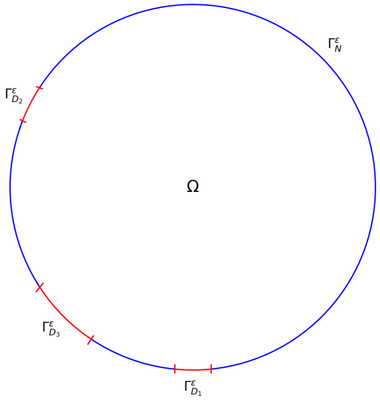
$$\begin{cases} -\Delta \nu_\varepsilon = \lambda_\varepsilon \nu_\varepsilon & \text{in } \Omega \\ \partial_n \nu_\varepsilon = 0 & \text{on } \Gamma_N^\varepsilon \\ \nu_\varepsilon = 0 & \text{on } \Gamma_D^\varepsilon \end{cases}$$

Furthermore, ν_ε is continuous on $\overline{\Omega}$, and $\partial_n \nu_\varepsilon$ is a measure on $\partial\Omega$.

The QSD as an eigenvalue problem

We want to find the QSD ν_ε

$$\begin{cases} -\Delta \nu_\varepsilon = \lambda_\varepsilon \nu_\varepsilon & \text{in } \Omega \\ \partial_n \nu_\varepsilon = 0 & \text{on } \Gamma_N^\varepsilon \\ \nu_\varepsilon = 0 & \text{on } \Gamma_D^\varepsilon \end{cases}$$



But thanks to [2]:

It depends on the angle α between Γ_N^ε and Γ_D^ε:

$$\alpha = \pi \Rightarrow \partial_n \nu_\varepsilon \notin L^2(\partial\Omega)$$

[2] Jakab, Mitrea and Mitrea, *Indiana University Mathematics Journal*, (2009)

Why modifying the domain?

We want to find the QSD ν_ε

$$\begin{cases} -\Delta \nu_\varepsilon = \lambda_\varepsilon \nu_\varepsilon & \text{in } \Omega_\varepsilon \\ \partial_n \nu_\varepsilon = 0 & \text{on } \Gamma_N^\varepsilon \\ \nu_\varepsilon = 0 & \text{on } \Gamma_D^\varepsilon \end{cases}$$

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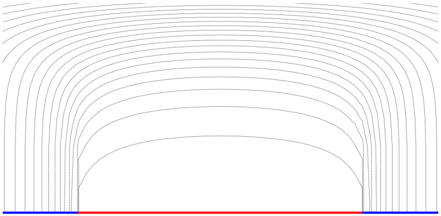
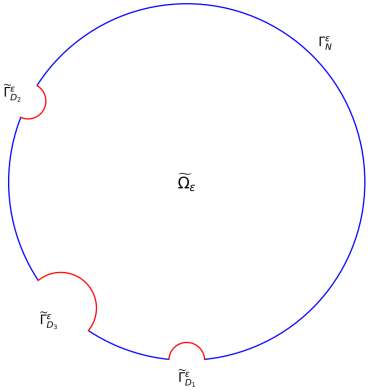


Figure: Level curves of the solution ν_ε near a flat hole.

[2] Jakab, Mitrea and Mitrea, *Indiana University Mathematics Journal*, (2009)



Similar eigenvalue problem:

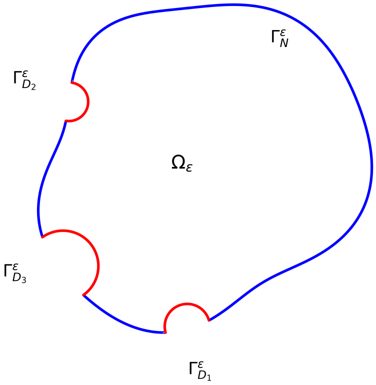
$$\begin{cases} -\Delta \nu_\varepsilon = \lambda_\varepsilon \nu_\varepsilon & \text{in } \Omega_\varepsilon \\ \partial_n \nu_\varepsilon = 0 & \text{on } \Gamma_N^\varepsilon \\ \nu_\varepsilon = 0 & \text{on } \widetilde{\Gamma}_D^\varepsilon \end{cases} \tag{3}$$

N holes of radius $r_\varepsilon^{(i)}$ centered at $x^{(i)} \in \partial\Omega$

Domain $\Omega_\varepsilon = \Omega \backslash \overline{\cup_{i=1}^N B(x^{(i)}, r_\varepsilon^{(i)})}$

New holes: $\widetilde{\Gamma}_{D_i}^\varepsilon = \partial B(x^{(i)}, r_\varepsilon^{(i)}) \cap \Omega$

A more regular narrow escape problem



Similar eigenvalue problem:

$$\begin{cases} -\Delta \nu_\varepsilon = \lambda_\varepsilon \nu_\varepsilon & \text{in } \Omega_\varepsilon \\ \partial_n \nu_\varepsilon = 0 & \text{on } \Gamma_N^\varepsilon \\ \nu_\varepsilon = 0 & \text{on } \Gamma_D^\varepsilon \end{cases}$$

Previous work: Asymptotic scaling for the disk and the ball [3]

My PhD work: Asymptotic scaling for general domains in $d \geq 2$ dimensions

[3] Lelièvre, Rachid and Stoltz, *preprint* (2024)

What does ν_ε look like?

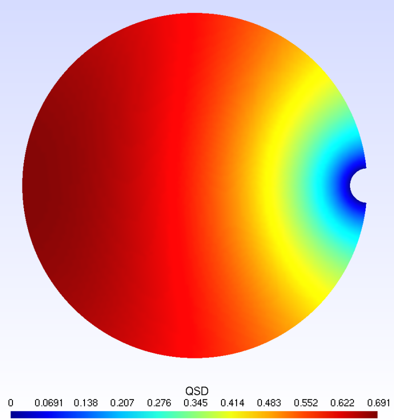


Figure: Dimension 2: Circle

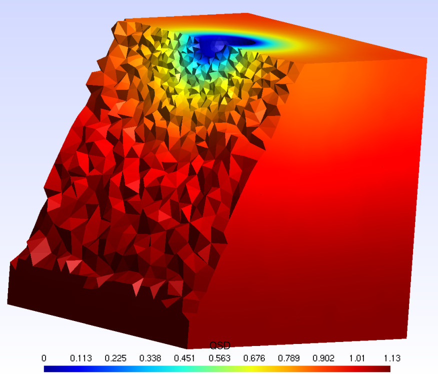


Figure: Dimension 3: Cube

Heuristic argument for 1 hole

Goal: We want to find an approximation φ^ε of the QSD ν_ε

Comparison to the Neumann problem: When $\varepsilon \to 0$, it holds $|\Gamma^\varepsilon_{\mathrm{D}}| \to 0$ so we expect $\lambda_\varepsilon \to 0$ and $\nu_\varepsilon \to \mathrm{cst}$ and the influence of the hole is local

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This motivates looking for a quasimode of the form

$$\varphi^\varepsilon = 1 + K_\varepsilon f,$$

with $K_\varepsilon \to 0$ when $\varepsilon \to 0$ and f a solution of, for $C > 0$,

$$\begin{cases} -\Delta f = C & \text{on } \Omega \\ \partial_n f = 0 & \text{on } \partial\Omega \setminus \{x^{(h)}\} \end{cases}$$

Discussion on the Ansatz for 1 hole

ν_ε is solution of

$$\begin{cases} -\Delta\nu_\varepsilon = \lambda_\varepsilon\nu_\varepsilon & \text{in } \Omega_\varepsilon \\ \partial_n\nu_\varepsilon = 0 & \text{on } \Gamma_N^\varepsilon \\ \nu_\varepsilon = 0 & \text{on } \Gamma_D^\varepsilon \end{cases}$$

The quasimode

$\varphi_\varepsilon = 1 + K_\varepsilon f$ where f is solution of

$$\begin{cases} -\Delta f = C & \text{on } \Omega \\ \partial_n f = 0 & \text{on } \partial\Omega \setminus \{x^{(h)}\} \end{cases}$$

The quasimode φ_ε verifies

$$\begin{cases} -\Delta\varphi_\varepsilon = CK_\varepsilon = CK_\varepsilon\varphi_\varepsilon - CK_\varepsilon^2 f & \text{on } \Omega_\varepsilon \\ \partial_n\varphi_\varepsilon = 0 & \text{on } \Gamma_N^\varepsilon \\ \varphi_\varepsilon = 1 + K_\varepsilon f & \text{on } \Gamma_D^\varepsilon \end{cases}$$

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$\Rightarrow$ The constant CK_ε is an approximation to λ_ε

$\Rightarrow$ The function f must verify

- $K_\varepsilon f = \mathrm{o}(|\varphi_\varepsilon|)$ on $L^2(\Omega_\varepsilon)$
- $K_\varepsilon f = \mathrm{o}(1)$ on $L^\infty(\Gamma_D^\varepsilon)$

Construction of the quasimode

By the compatibility equation

$$-C|\Omega| = \int_{\Omega} \Delta f = \int_{\partial\Omega} \partial_n f$$

But $\partial_n f = 0$ on $\partial\Omega \backslash \{x^{(h)}\}$, so $\partial_n f$ is not a function but a distribution. A part of my PhD is to have local expansion around the singularity of the solution of in the weak $W^{1,p}(\Omega)$ sense:

$$\begin{cases} -\Delta f = C & \text{in } \Omega \\ \partial_n f = -C|\Omega|\delta_{x^{(h)}} & \text{on } \partial\Omega \end{cases}$$

with $f \in \left\{ u \in W^{1,p}(\Omega),\ -\Delta u = C \in L^p(\Omega), \partial_n u = -C|\Omega|\delta_{x^{(h)}} \in W^{-\frac{1}{p},p}(\Omega) \right\}$

Construction of the quasimode

Explicit expressions for simple geometries [1], ex: the unit sphere with one hole on $x^{(h)}$

$$\varphi_\varepsilon(x) = 1 + \frac{1}{\sqrt{3\pi}} K_\varepsilon^{(h)} \left[\frac{1}{|x - x^{(h)}|} - \frac{1}{2} \log \left(1 - x \cdot x^{(h)} + \left| x - x^{(h)} \right| \right) + \frac{|x|^2}{4} \right]$$

But in the general case, we have been able to do expansions around the singularity only in the following settings:

- **dimension** 2, any smooth domain for flat holes
- **dimension** 2 **and above**, any C^2 domain, locally smooth, with circular holes
- **dimension** 3, any smooth domain for slit-like holes

Properties of the quasimode (N holes)

For N holes of radius $r_\varepsilon^{(k)}$ centered at $x^{(k)}$, denote

$$K_\varepsilon^k = \begin{cases} -\dfrac{1}{\log(r_k^\varepsilon)} & \text{if } d = 2 \\ (r_k^\varepsilon)^{d-2} & \text{if } d \geq 3 \end{cases} \qquad \text{and} \qquad \mathcal{E} : x \mapsto \begin{cases} x & \text{if } d = 2 \\ x\,|\log(x)| & \text{if } d = 3 \\ x^{1/(d-2)} & \text{if } d \geq 4 \end{cases}$$

Lemma [Quasimode properties]

Let φ_ε be the quasimode defined by $\varphi_\varepsilon = 1 + \sum_{k=1}^N K_\varepsilon^k f_k$. Then φ_ε is smooth on Ω_ε and it verifies

- $-\Delta \varphi_\varepsilon = C_{d,\Omega} > 0$ on Ω_ε
- $\partial_n \varphi_\varepsilon = 0$ on $\partial\Omega_\varepsilon$
- $\left|(\nu_\varepsilon, \varphi_\varepsilon)_{\Omega_\varepsilon} - 1\right| = \mathrm{O}\left(\mathcal{E}(\sum_{k=1}^N K_\varepsilon^k)\right)$
- $\|\varphi_\varepsilon\|_{L^\infty(\Gamma_\mathcal{D}^\varepsilon)} = \mathrm{O}\left(\mathcal{E}(\sum_{k=1}^N K_\varepsilon^k)\right)$

Attention: φ_ε is not in the form/operator domain

Results for N exit holes and $d \geq 2$

Theorem (Eigenvalue) [C., Lelièvre, Normand, Stoltz, Vaes]

The mean exit time of the process $(X_t)_{t\geq 0}$, when $X_0 \sim \nu_\varepsilon$ is given by $\mathbb{E}_{\nu_\varepsilon}[\tau_\varepsilon] = \frac{1}{\lambda_\varepsilon}$, where

$$\lambda_\varepsilon = C_{d,\Omega}\overline{K}_\varepsilon + \mathrm{O}\Big(\overline{K}_\varepsilon\,\mathcal{E}(\overline{K}_\varepsilon)\Big)$$

with $\overline{K}_\varepsilon = \sum_{k=1}^N K_\varepsilon^k$ and $C_{d,\Omega} > 0$ defined as

$$C_{d,\Omega} = \frac{\max\{d-2,1\}}{2|\Omega|}\,|\partial B(0,1)|$$

Results for 1 exit hole and $d \geq 2$

Theorem (Eigenvalue) [C., Lelièvre, Normand, Stoltz, Vaes]

Consider only one hole of radius r_ε. Then

$$\lambda_\varepsilon = \Big(\mathbb{E}[\tau_\varepsilon]\Big)^{-1} = \begin{cases} C_{2,\Omega}\,(\log(r_\varepsilon))^{-1} & + \mathrm{O}\left([\log(r_\varepsilon)]^{-2}\right), & \text{for } d = 2 \\ C_{3,\Omega}\,r_\varepsilon & + \mathrm{O}({r_\varepsilon}^2 \log(r_\varepsilon)), & \text{for } d = 3 \\ C_{d,\Omega}\,r_\varepsilon^{d-2} & + \mathrm{O}(r_\varepsilon^{d-1}), & \text{for } d \geq 4 \end{cases}$$

with $C_{d,\Omega} > 0$ defined as

$$C_{d,\Omega} = \frac{\max\{d-2, 1\}}{2|\Omega|}\,|\partial B(0,1)|$$

Proof

With Green identity

$$\lambda_\varepsilon = \frac{(\varphi_\varepsilon,\, -\Delta \nu_\varepsilon)_{\Omega_\varepsilon}}{(\varphi_\varepsilon,\, \nu_\varepsilon)_{\Omega_\varepsilon}} = \frac{(-\Delta \varphi_\varepsilon,\, \nu_\varepsilon)_{\Omega_\varepsilon} + (\partial_n \varphi_\varepsilon,\, \nu_\varepsilon)_{\partial\Omega_\varepsilon} - (\varphi_\varepsilon,\, \partial_n \nu_\varepsilon)_{\partial\Omega_\varepsilon}}{(\varphi_\varepsilon,\, \nu_\varepsilon)_{\Omega_\varepsilon}}$$

$$= \frac{C_{3,\Omega}\overline{K}_\varepsilon + (\varphi_\varepsilon,\, \partial_n \nu_\varepsilon)_{\Gamma_D^\varepsilon}}{(\varphi_\varepsilon,\, \nu_\varepsilon)_{\Omega_\varepsilon}}$$

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$$= \frac{C_{3,\Omega}\overline{K}_\varepsilon + (\varphi_\varepsilon,\, \partial_n\nu_\varepsilon)_{\Gamma_D^\varepsilon}}{(\varphi_\varepsilon,\, \nu_\varepsilon)_{\Omega_\varepsilon}}$$

Since $\|\varphi_\varepsilon\|_{L^\infty(\Gamma_{\mathcal{D}}^\varepsilon)} = \mathrm{O}\left(\mathcal{E}(\overline{K}_\varepsilon)\right)$, one has $(\varphi_\varepsilon,\, \partial_n\nu_\varepsilon)_{\Gamma_D^\varepsilon} = \lambda_\varepsilon \mathrm{O}\left(\mathcal{E}(\overline{K}_\varepsilon)\right)$

Proof

With Green identity

$$\lambda_\varepsilon = \frac{(\varphi_\varepsilon,\, -\Delta\nu_\varepsilon)_{\Omega_\varepsilon}}{(\varphi_\varepsilon,\, \nu_\varepsilon)_{\Omega_\varepsilon}} = \frac{(-\Delta\varphi_\varepsilon,\, \nu_\varepsilon)_{\Omega_\varepsilon} + (\partial_n\varphi_\varepsilon,\, \nu_\varepsilon)_{\partial\Omega_\varepsilon} - (\varphi_\varepsilon,\, \partial_n\nu_\varepsilon)_{\partial\Omega_\varepsilon}}{(\varphi_\varepsilon,\, \nu_\varepsilon)_{\Omega_\varepsilon}}$$

$$= \frac{C_{3,\Omega}\overline{K}_\varepsilon + (\varphi_\varepsilon,\, \partial_n\nu_\varepsilon)_{\Gamma_D^\varepsilon}}{(\varphi_\varepsilon,\, \nu_\varepsilon)_{\Omega_\varepsilon}}$$

Since $\|\varphi_\varepsilon\|_{L^\infty(\Gamma_\mathcal{D}^\varepsilon)} = \mathrm{O}\left(\mathcal{E}(\overline{K}_\varepsilon)\right)$, one has $(\varphi_\varepsilon,\, \partial_n\nu_\varepsilon)_{\Gamma_D^\varepsilon} = \lambda_\varepsilon \mathrm{O}\left(\mathcal{E}(\overline{K}_\varepsilon)\right)$
Furthermore $\left|(\nu_\varepsilon,\, \varphi_\varepsilon)_{\Omega_\varepsilon} - 1\right| = \mathrm{O}\left(\mathcal{E}(\overline{K}_\varepsilon)\right)$, thus

$$\lambda_\varepsilon = C_{3,\Omega}\overline{K}_\varepsilon + \lambda_\varepsilon \mathrm{O}\left(\mathcal{E}(\overline{K}_\varepsilon)\right) + \mathrm{O}\left(\overline{K}_\varepsilon\mathcal{E}(\overline{K}_\varepsilon)\right)$$

Proof

With Green identity

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Then the result follows

Exit point for 2 exit holes and $d \geq 2$

Theorem (Exit hole distribution)[C., Lelièvre, Normand, Stoltz, Vaes]

The probability to exit through hole $i \in \{1, \dots, N\}$ of the process (X_t) initialised at ν_ε is given by:

$$\mathbb{P}_\varepsilon(X_{\tau_\varepsilon} \in \Gamma^\varepsilon_{D_i}) = \frac{\underline{K^i_\varepsilon}}{\overline{K_\varepsilon}} + \mathrm{O}\Big(\mathcal{E}(\overline{K_\varepsilon})\Big)$$

Example: with $N = 2$, scaling as $r^{(1)}_\varepsilon = \varepsilon$ and $r^{(2)}_\varepsilon = 2\varepsilon$

- in dimension 2, $\mathbb{P}_{\nu_\varepsilon}(X_{\tau_\varepsilon} \in \Gamma^\varepsilon_{D_1}) \approx \dfrac{1}{2}$ equal to $\mathbb{P}_{\nu_\varepsilon}(X_{\tau_\varepsilon} \in \Gamma^\varepsilon_{D_2})$
- in dimension 3, $\mathbb{P}_{\nu_\varepsilon}(X_{\tau_\varepsilon} \in \Gamma^\varepsilon_{D_1}) \approx \dfrac{1}{3}$ twice smaller than $\mathbb{P}_{\nu_\varepsilon}(X_{\tau_\varepsilon} \in \Gamma^\varepsilon_{D_2})$
- in dimension 4, $\mathbb{P}_{\nu_\varepsilon}(X_{\tau_\varepsilon} \in \Gamma^\varepsilon_{D_1}) \approx \dfrac{1}{5}$ four times smaller than $\mathbb{P}_{\nu_\varepsilon}(X_{\tau_\varepsilon} \in \Gamma^\varepsilon_{D_2})$

Exit point for 2 exit holes and $d \geq 2$

Theorem (Exit hole distribution)[C., Lelièvre, Normand, Stoltz, Vaes]

The probability to exit through hole $i \in \{1, \dots, N\}$ of the process (X_t) initialised at ν_ε is given by:

$$\mathbb{P}_\varepsilon(X_{\tau_\varepsilon} \in \Gamma^\varepsilon_{D_i}) = \frac{K^i_\varepsilon}{\overline{K_\varepsilon}} + \mathrm{O}\Big(\mathcal{E}(\overline{K_\varepsilon})\Big)$$

Idea of proof:

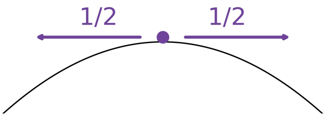
$$\mathbb{P}_\varepsilon(X_{\tau_\varepsilon} \in \Gamma^\varepsilon_{D_k}) = \frac{-1}{\lambda_\varepsilon}(\partial_n \nu, 1)_{\Gamma^\varepsilon_{D_k}} = \frac{-1}{\lambda_\varepsilon}\left(\partial_n \nu, \varphi_\varepsilon^{-k}\right)_{\Gamma^\varepsilon_{D_k}}$$

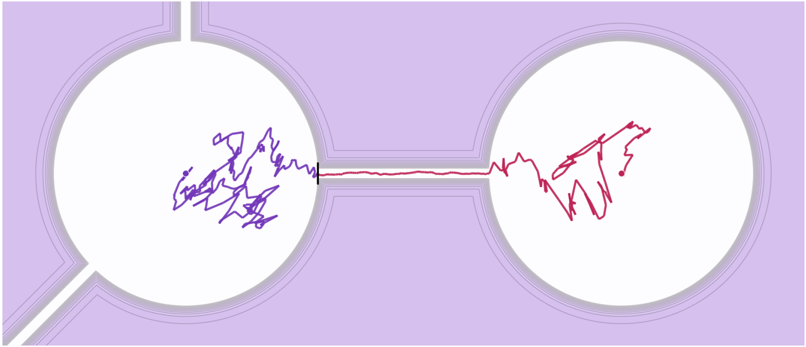
where $\varphi_\varepsilon^{-k} = 1 + \sum_{i \neq k} K^i_\varepsilon f_i$ is the quasimode associated to the problem with $N-1$ holes, without the hole k

In the energetic world, transition time is

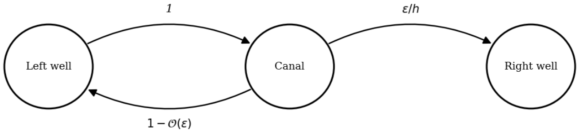
twice the exit time

Is it the case in the entropic world?





When we are in the narrow canal, we are studying a one-dimensional Brownian motion. The probability to come back to the left well is ε/h, where h is the length of the canal.



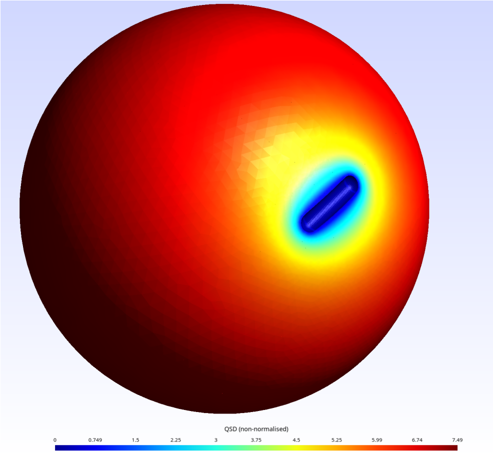
Rough estimate $\mathbb{E}[\tau_{\text{transition}}] \approx \frac{h}{\varepsilon}$

A bit of literature

Holcman and Schuss, *The Narrow Escape Problem*, (2014)
Li, *A new model for solving narrow escape problem in domain with long neck*, (2014)
Hsu, *Asymptotic analysis on narrow tubes: narrow escape problems and diffusion processes*, (2025)

Conclusion

- The narrow escape is a toy model of metastability of entropic origin
- With our approach we can solve it for any smooth domain in any dimension
- We get the scaling of the escape time and the law of exit hole



Future work:

- Study the influence of the hole geometry on the escape event → the slit
- Study the transition time in the narrow canal → the narrow transition problem