Narrow escape problem and Quasi-stationary distribution

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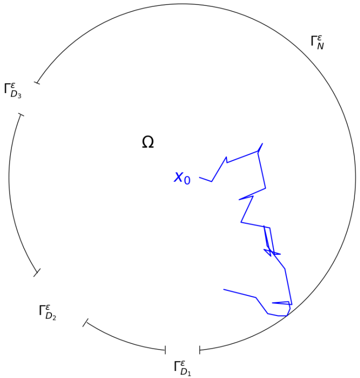
Rare event : metastability induced by entropic barriers ($\neq$ energetic)



Setting:

- Domain Ω with holes $\Gamma_{D_i}^\varepsilon$ and reflecting boundary Γ_N^ε
- A Brownian motion starting at x_0 taking a long time to exit $\tau_\varepsilon = \inf\{t \geq 0 \,|\, X_t \notin \overline{\Omega}\}$

Goal: In the limit of small holes $\varepsilon \to 0$:

- Distribution of the escape time τ_ε
- The exit point distribution X_{τ_ε}

[1] Introduced by Holcman and Schuss (2004), then large numbers of contributors: Ammari, Bénichou, Chen, Chevalier, Cheviakov, Friedman, Grebenkov, Singer, Straube, Voituriez, Ward...

Definition

For $x \in \Omega,\ \nu_\varepsilon(x) = \lim_{t\to\infty} \mathbb{P}(X_t = x \,|\, t < \tau_\varepsilon)$

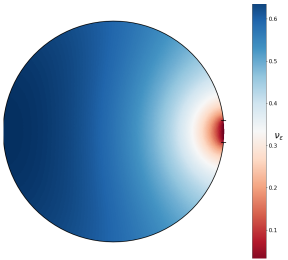


Figure: ν_ε for a circle with 1 hole. The data was obtained through finite element methods.

Why is it natural?

- Counterpart of the stationary distribution for metastable systems

Why is it useful?

- If $X_0 \sim \nu_\varepsilon$,

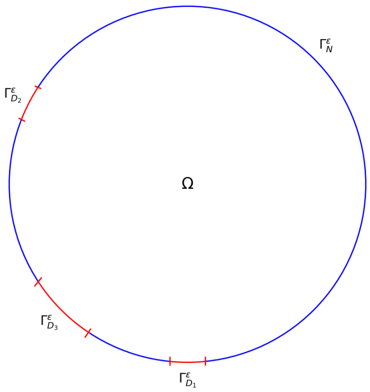
  $\forall t > 0,\ \mathcal{L}(X_t \,|\, t < \tau_\varepsilon) = \nu_\varepsilon$

  $\tau_\varepsilon \sim \mathrm{Exp}(\lambda_\varepsilon)$ independent of X_{τ_ε}

  $X_{\tau_\varepsilon} \sim \partial_n \nu_\varepsilon$
- Markov jump process [2]

[2] Di Gesù, Lelièvre, Le Peutrec and Nectoux, *Faraday Discussion*, (2016)

The orignal narrow escape problem



Eigenvalue problem:

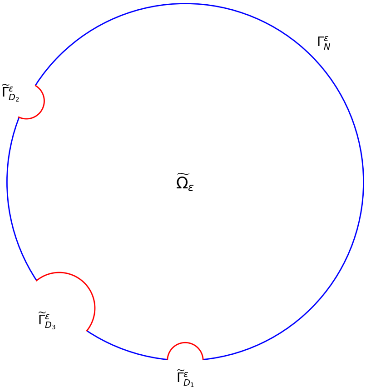
$$\begin{cases} -\Delta \nu_\varepsilon = \lambda_\varepsilon \nu_\varepsilon & \text{in } \Omega \\ \partial_n \nu_\varepsilon = 0 & \text{on } \Gamma_N^\varepsilon \\ \nu_\varepsilon = 0 & \text{on } \Gamma_{D_i}^\varepsilon \end{cases}$$

Flat angle between Γ_N^ε and $\Gamma_{D_i}^\varepsilon$: $\partial_n \nu_\varepsilon(x) \underset{x \to \partial \Gamma_{D_i}^\varepsilon}{\longrightarrow} +\infty$

This singularity makes ν_ε hard to approximate analytically $\Rightarrow$ **geometry** change

A more regular narrow escape problem

Remove the singularity by **modifying** the domain without **altering** too much τ_ε and X_{τ_ε}.



Similar eigenvalue problem:

$$\begin{cases} -\Delta \nu_\varepsilon = \lambda_\varepsilon \nu_\varepsilon & \text{in } \widetilde{\Omega}_\varepsilon \\ \partial_n \nu_\varepsilon = 0 & \text{on } \Gamma_N^\varepsilon \\ \nu_\varepsilon = 0 & \text{on } \widetilde{\Gamma}_{D_i}^\varepsilon \end{cases}$$

Previous work: Exact solution for the disk and the ball [3]

My PhD work: Asymptotic solution for general domains in 2 and 3 dimensions

[3] Lelièvre, Rachid and Stoltz, *preprint* (2024)